Observation of intermingled basins in coupled oscillators exhibiting synchronized chaos

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Recent work has shown that chaotic systems possessing invariant manifolds of lower dimension than that of the full phase space can exhibit an interesting class of phenomena including riddled basins, intermingled basins and on-off intermittency. In particular, if a physical system is characterized by intermingled basins, no finite computation can determine the fate of an exactly given initial condition. In other words, the dynamics is uncomputable. In this work we wish to show that intermingled basins can be easily realized in the context of coupled oscillators and synchronized chaos. This opens the possibility of investigating these phenomena in laboratory experiments. [S1063-651X(96)08809-5]

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Consider an m-dimensional dynamical system that possesses an n-dimensional invariant manifold where $n<m$. Here by invariant we mean that a trajectory initialized in the manifold stays there for all time. Suppose that the dynamics restricted to the invariant manifold has two or more chaotic attractors. Depending on how small perturbations transversal to the manifold behave under the equations of motion, these restricted attractors may or may not be attractors for the full phase space. Recently, it has been shown that, when these restricted attractors are also attractors for the full system, under certain conditions, their basins of attraction are intermingled [1–3]. By this we mean that the basins are so finely mixed that every point in the basin of one attractor has points from the basins of other attractors arbitrarily nearby, and vice versa. It is further pointed out in Ref. [3] that when this type of basin structure occurs, the system's dynamics become qualitatively undecidable [4]. That is, no finite computation can determine the fate of a typical initial condition even if that initial condition is given with infinite precision.

Much of our current understanding of the characteristics of intermingled basins is obtained through the study of discrete maps [1]. Sommerer and Ott [3] examine a physical model displaying intermingled basins in which a particle moves under periodic forcing in a two-dimensional potential. A similar physical system with multiple invariant manifolds is considered by Lai and Grebogi in Ref. [2]. In this paper we report our observation of intermingled basins in systems of coupled nonlinear oscillations exhibiting synchronized chaos. Here the synchronization manifold is the requisite lower dimensional invariant manifold. Given that the literature is replete with examples of coupled physical devices (e.g., nonlinear circuits) displaying synchronized chaotic behavior, we thus believe that our result helps to pave the way for future experimental investigations of intermingled basins and related phenomena. We illustrate our main point using two examples, one a coupled Duffing oscillator and the other a coupled map. The reader is referred to Refs. [5,6] for a sample of works dealing with other topics such as riddled basins and on-off intermittency that also arise in systems having invariant manifolds of lower dimensions.

Example 1: The Coupled Duffing Oscillator. We express a single periodically driven Duffing oscillator as

$$\ddot{x}+\mu\dot{x}+\alpha(x-x^3)=A\ \sin(\omega t). \tag{1}$$

For $\mu=0.632$, $\alpha=-4$, $A=1.011$, and $\omega=2.1235$, this equation has two chaotic attractors [3], one lying in the region $x>0$ which we denote A^+, and the other in the region $x<0$ which we denote A^-. The coupled Duffing oscillator we treat in this example is written as

$$\ddot{x}+\mu\dot{x}+\alpha(x-x^3)+p(x-y)+q(x^2-y^2)=A\ \sin(\omega t), \tag{2}$$

$$\ddot{y}+\mu\dot{y}+\alpha(y-y^3)+p(y-x)+q(y^2-x^2)=A\ \sin(\omega t). \tag{3}$$

Our experience indicates that it is essential to include nonlinear coupling terms in this model for the occurrence of intermingled basins. In what follows we fix $q=0.005$ and vary p as a parameter. All other parameters have values as given above. This coupled oscillator is a five dimensional system. Strobing the equations at times $t_n=n2\pi/\omega$, we obtain a four dimensional Poincaré map ($m=4$) for x, $\dot{x}$, y, and $\dot{y}$. We henceforth carry out our discussion in terms of this Poincaré map.

In the absence of coupling the two oscillators are identical. Note that if $x(t)=y(t)$ is plugged into Eqs. (2) and (3), the equations are satisfied, meaning that synchronization of chaos is possible for the system. The two dimensional synchronization manifold ($n=2$) is defined by $x=y$ and $\dot{x}=\dot{y}$, and is invariant under the dynamics. In the synchronization manifold the dynamics is identical to that of a single Duffing oscillator. That is, there are two chaotic attractors in this manifold denoted by A^+ and A^-. To determine whether the synchronization manifold is attracting, that is, to determine whether A^+ and A^- are attractors in the full phase space, we compute the largest transversal Lyapunov exponents as a function of p for both attractors. The result is displayed in Fig. 1, with the solid line for A^- and the dashed line for A^+. The values of p_1 and p_2 are determined to be $p_1=0.1946$ and $p_2=0.2103$. For $p>p_2$, both largest transversal Lyapunov exponents are negative, indicating that A^+ and A^- are global attractors (in the sense of Milnor [7]). Our numerical basin result in Fig. 2 strongly suggests that their

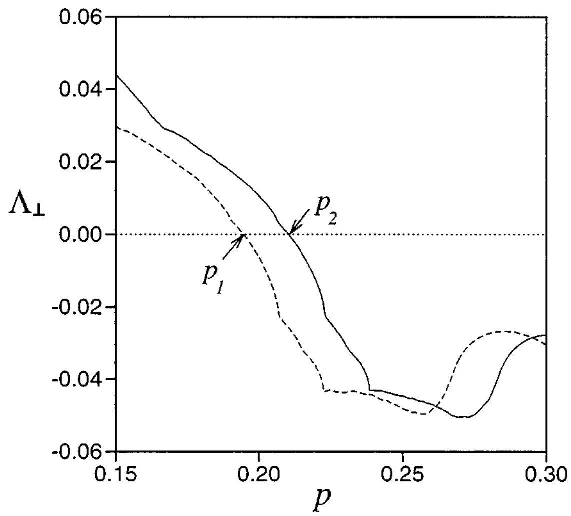

FIG. 1. The largest transversal Lyapunov exponent for A^- (solid line) and for A^+ (dashed line). The model is the coupled Duffing oscillator Eqs. (2) and (3).

basins of attraction are intermingled. Specifically, let $p=0.25$ and consider a two dimensional plane defined by $y=x+d$ and $\dot{y}=\dot{x}$. This plane is parallel to the synchronization manifold, $x=y$ and $\dot{x}=\dot{y}$, which is also a two dimensional plane. The distance between the two parallel planes is d. For Fig. 2 we use $d=0.01$. The horizontal axis is x and the vertical axis is $\dot{x}$. The initial conditions are placed on a uniform grid of 300×300 points. If a point in this grid goes to the attractor A^- after 400 periods of external driving, we plot a dot at the point, and if a point goes to the attractor A^+ we leave the point blank. Longer iteration times are also used in our numerical simulations, and they yield similar results. Since the plane used in Fig. 2 is rather close to the synchronization manifold, the basin structure in Fig. 2(a) roughly resembles that of a single Duffing oscillator [see Fig. 2(a) of Ref. [3]]. The important difference here is that both basins contain no solid regions. This is demonstrated using two successive enlargements in Figs. 2(b) and 2(c). At a given resolution one may find some regions showing just one color. Upon magnification, however, points belonging to the other basin begin to emerge in the region. This process appears to go on *ad infinitum*, implying that neither basin contains solid regions. This result leads us to the conjecture that the basins in Fig. 2 are intermingled.

Further evidence of intermingling is provided by examining the dynamics for $p<p_2$. Specifically, for $p_1<p<p_2$, the largest transversal Lyapunov exponent for A^- becomes positive, and A^+ remains the only global attractor of the system. A typical trajectory, before it finally settles on A^+, undergoes an intermittent transient process in the variable x-y, as illustrated in Fig. 3(a) where $p=0.195$. The key point here is contained in Fig. 3(b), where we see that the intermittent trajectory jumps back and forth between A^+ and A^- (a phenomenon we call communication), and comes arbitrarily close to both of them during the process. The average length of the transient gets longer as p gets closer to p_2 from below. This strongly indicates that as p is increased beyond p_2, from continuity, we can still expect that points arbitrarily

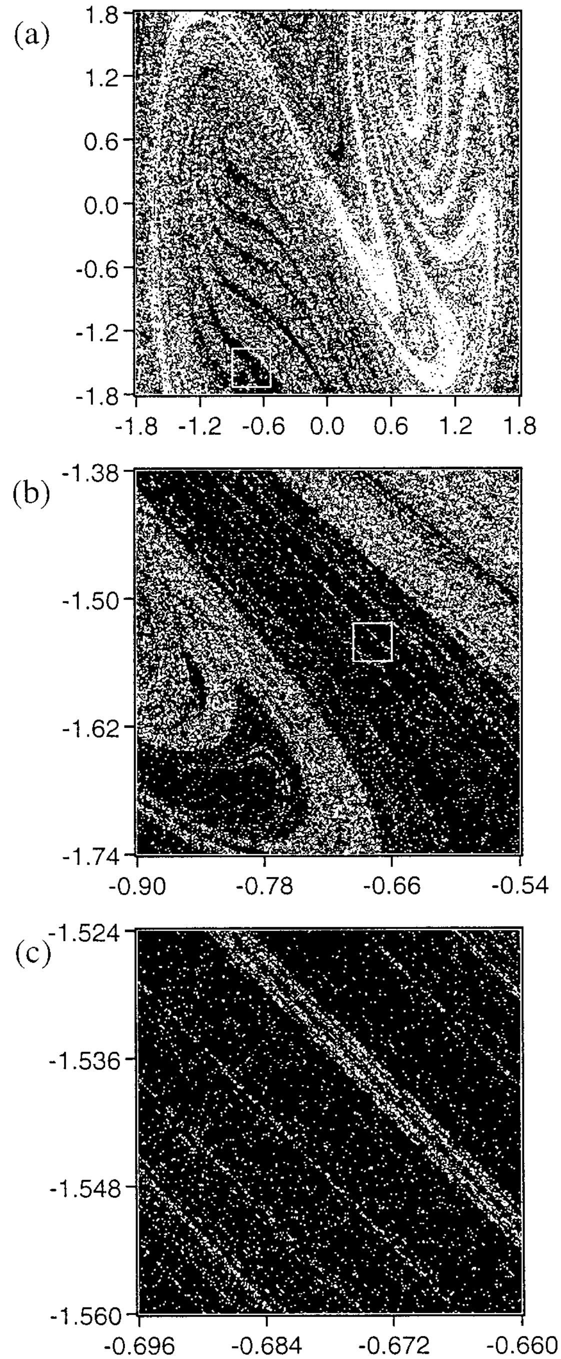

FIG. 2. The basin structures for the coupled Duffing oscillator. (b) is the enlargement of the marked box in (a), and (c) is the enlargement of the marked box in (b). The horizontal axis is x and the vertical axis is $\dot{x}$. Here $p=0.25>p_2$.

close to, say, A^- can move to A^+ as time evolves, and vice versa. This means that there are points which are arbitrarily close to one attractor actually belong to the basin of the other.

For $p<p_1$ both A^- and A^+ become transversally unstable. For p slightly less than p_1 we observe sustained on-off intermittency an example of which is shown in Fig. 4(a) for $p=0.19$. Again during the on-off intermittent process the trajectory communicates between A^+ and A^- [Fig. 4(b)].

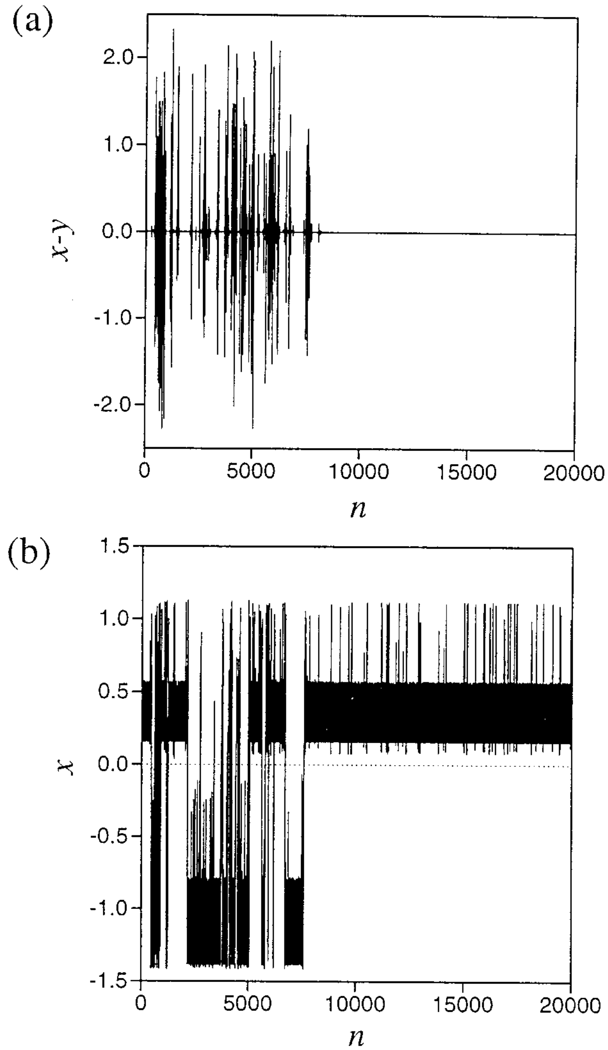

FIG. 3. On-off intermittent transient for the coupled Duffing oscillator for $p_1 < p = 0.195 < p_2$ (a). Note the communication behavior between A^+ and A^- by the trajectory x in (b).

Note that, at $p = 0.19$, the largest transversal Lyapunov exponent for A^+ is only slightly positive and is much smaller than that for A^-. This means that A^+ is a much weaker transversal repeller than A^-. Reflected in the temporal dynamics, the trajectory tends to stay longer near A^+ than near A^-.

The findings we have made thus far can be summarized as follows. For $p > p_2$ we observe intermingled basins (Fig. 2). The interval of $p_1 < p < p_2$ is characterized by on-off intermittent transients with A^+ as the only final attractor (Fig. 3). When p is decreased below p_1, both A^+ and A^- are no longer global attractors, and the system exhibits sustained on-off intermittency. Moreover, the on-off intermittent temporal behavior has the character that the intermittent trajectory communicates between A^+ and A^-. In example 2 below we will further illustrate the significance of this type of communication phenomenon in relation to the existence of intermingled basins.

In the terminology of Ref. [8], the transition at $p = p_1$ is a nonhysteretic blowout bifurcation, and the transition at $p = p_2$ is a hysteretic blowout bifurcation. It is interesting to

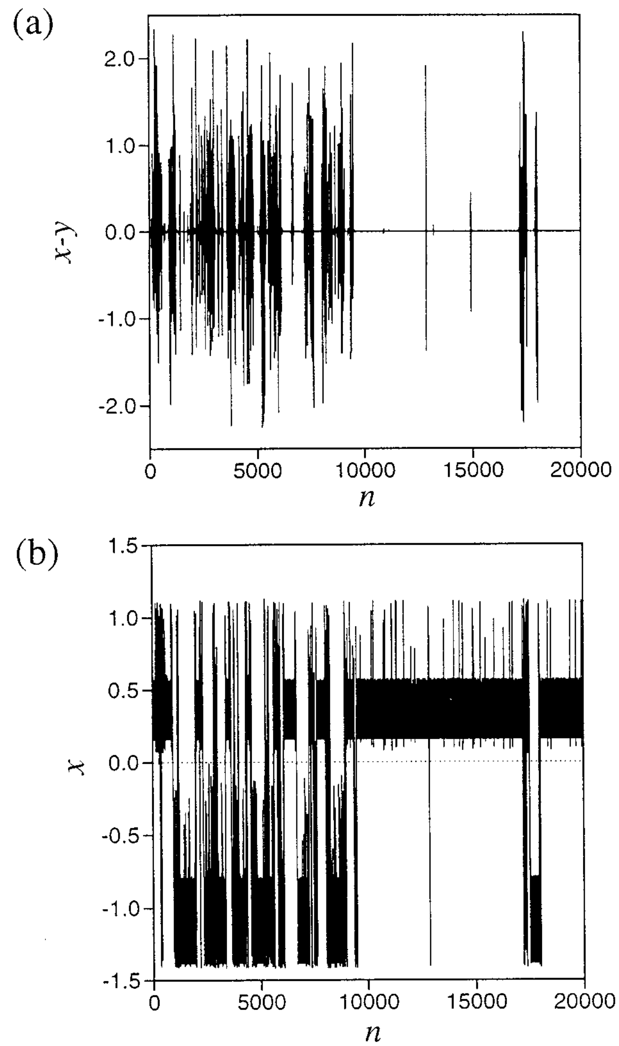

FIG. 4. Sustained on-off intermittency for the coupled Duffing oscillator for $p = 0.19 < p_1$ (a). Note the communication behavior between A^+ and A^- by the trajectory x in (b).

note that by varying a single parameter p one can achieve both bifurcations in the same system.

Example 2. Now we consider a coupled map system expressed as

$$x_{n+1} = f(x_n) + \epsilon[f(y_n) - f(x_n)] - p[f(y_n)^3 - f(x_n)^3], \quad (4)$$

$$y_{n+1} = f(y_n) + \epsilon[f(x_n) - f(y_n)] - p[f(x_n)^3 - f(y_n)^3], \quad (5)$$

where $f(x) = ax(1-x^2)e^{-x^2}$. The phase space is the two dimensional plane $(m=2)$. For $a = 3.4$ the individual map $x_{n+1} = f(x_n)$ has two chaotic attractors, one in the region $x > 0$ which we denote A^+, and the other in the region $x < 0$ which we denote A^-. These two attractors are symmetric with respect to $x = 0$. (This symmetry is not relevant for the existence of intermingled basins.) Clearly, if $x_n = y_n$ is plugged into Eqs. (4) and (5), the equations are satisfied, meaning that we can have synchronized chaos. The synchronization manifold, defined by $x = y$, is one dimensional $(n=1)$, and is invariant under the dynamics. Whether A^+

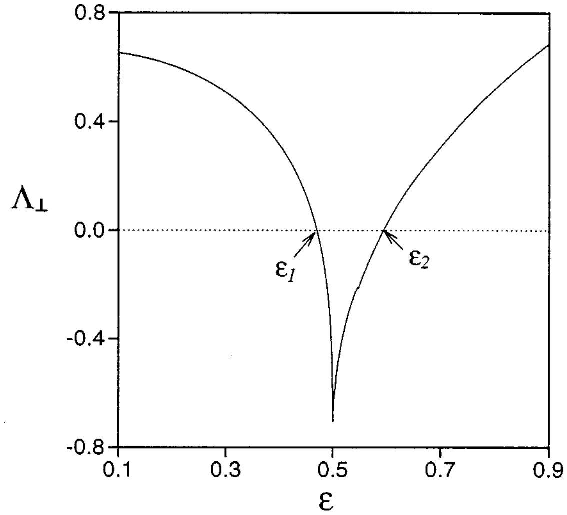

FIG. 5. The largest transversal Lyapunov exponent for A^+ and A^-. The model is the coupled map, Eqs. (4) and (5), and the two parameters are related by $p=\epsilon$.

and A^- are also attractors for the full two dimensional space is determined by their transversal Lyapunov exponents.

We present the transversal Lyapunov exponents as a function of ϵ in Fig. 5. Here we let $p=\epsilon$. Since the coupling used in Eqs. (4) and (5) preserves the symmetry between A^+ and A^-, the two transversal Lyapunov exponents are identical. The two transition points ϵ_1 and ϵ_2 are found to be at $\epsilon_1=0.4701$ and $\epsilon_2=0.5940$. For $\epsilon_1<\epsilon<\epsilon_2$, both A^+ and A^- are global attractors. In Fig. 6 we present evidence supporting the claim that their basins of attraction are intermingled. Here $\epsilon=0.48$. To produce this figure we use a 400×400 uniform grid. If an initial condition on the grid, after 10 000 iterations, goes to A^-, we plot a dot at the point, and if an initial condition goes to A^+ we leave the point blank. The result is a finely mixed structure of black and white points. Although not included in this paper, magnifications of any regions in the picture show qualitatively the same result. Further supporting evidence of intermingling is

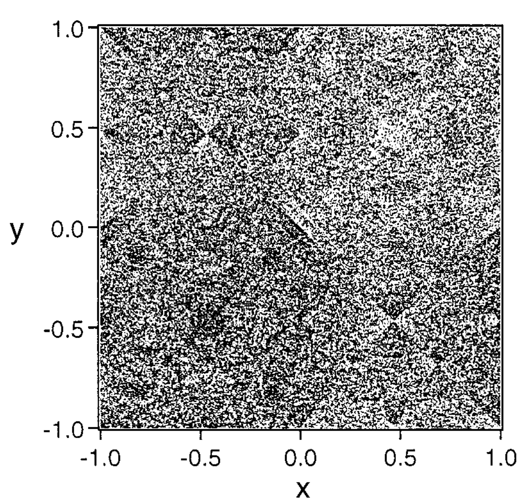

FIG. 6. The basin structure for the coupled map. The intermingled character of the basins is apparent. Here $\epsilon_1<\epsilon=p=0.48<\epsilon_2$.

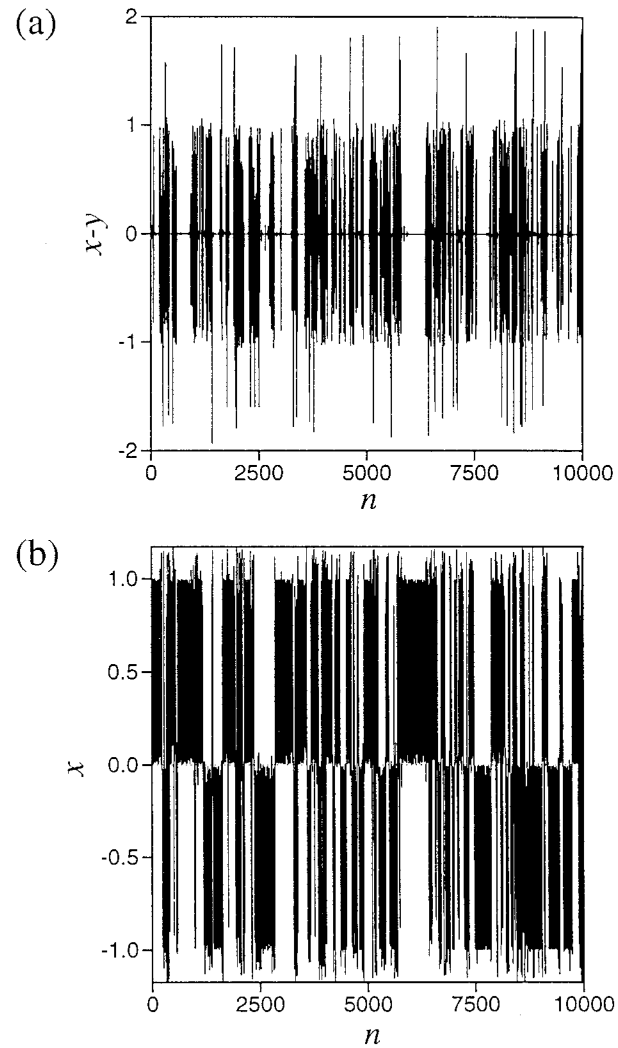

FIG. 7. Sustained on-off intermittency for the coupled map for $\epsilon=p=0.46<\epsilon_1$ (a). Again note the communication behavior between A^+ and A^- by the trajectory x in (b).

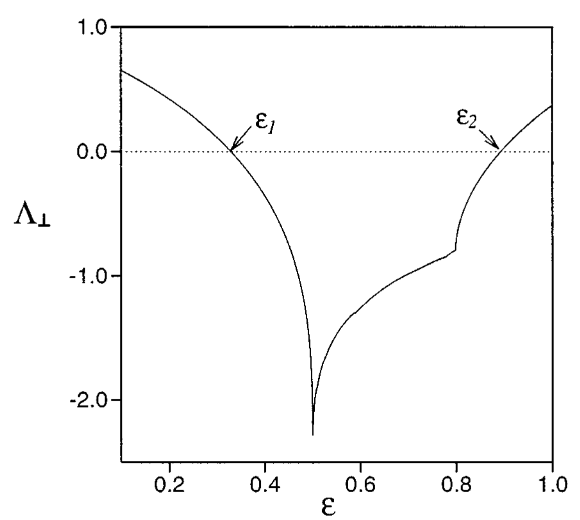

FIG. 8. The largest transversal Lyapunov exponent for A^+ and A^-. The model is the coupled map, Eqs. (4) and (5), and we set $p=0.1$.

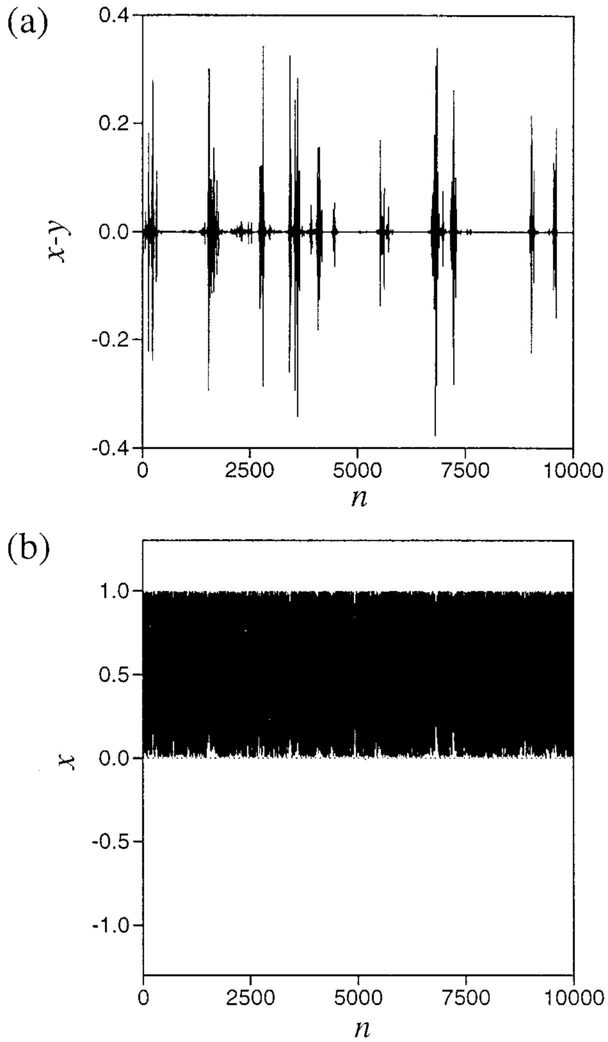

FIG. 9. Sustained on-off intermittency for the coupled map for $\epsilon=0.329<\epsilon_1$ and $p=0.1$ (a). Note the *lack* of communication behavior between A^+ and A^- by the trajectory x in (b).

furnished by the on-off intermittency behavior at $\epsilon=0.46$, which is slightly below ϵ_1. At this parameter value both A^+ and A^- are no longer global attractors. Figure 7 shows the temporal dynamics of the variable $x-y$ [Fig. 7(a)] and the variable x [Fig. 7(b)]. The on-off intermittent character is apparent in Fig. 7(a). More importantly, in Fig. 7(b) we observe the communication phenomenon by the trajectory between A^+ and A^-. As mentioned earlier, this is an essential ingredient for predicting the existence of intermingled basins

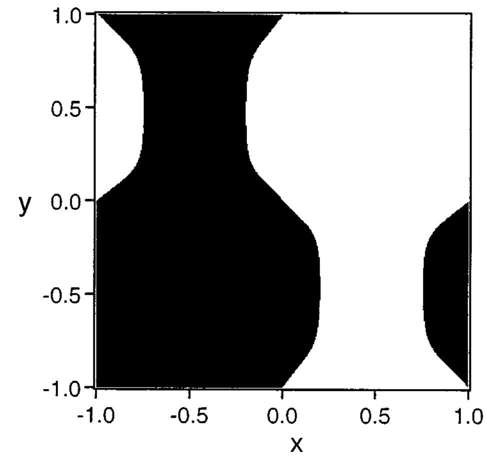

FIG. 10. The basin structure for the coupled map is not intermingled for the parameter setting $\epsilon_1<\epsilon=0.34<\epsilon_2$ and $p=0.1$. This is closely related to the fact that the on-off intermittent trajectory in Fig. 9 does not exhibit communication between the two restricted attractors A^+ and A^-.

for nearby parameter values. It can also serve as a cue in an experiment, since intermittency is relatively more direct to observe than intermingled basins.

Our next result is designed to further illustrate the significant of the communication phenomenon. Let $p=0.1$. The transversal Lyapunov exponent as a function of ϵ is shown in Fig. 8, where $\epsilon_1=0.3296$ and $\epsilon_2=0.8924$. For $\epsilon=0.329<\epsilon_1$, both A^+ and A^- are unstable in the transversal direction, and we observe sustained on-off intermittency in the variable $x-y$ [Fig. 9(a)]. But, as shown in Fig. 9(b), the intermittent trajectory stays only on the side of A^+. Due to the symmetry, intermittent trajectories started on the side of A^- will also remain on that side for all time. In other words, no communication takes place between A^+ and A^-. It is thus not surprising that when A^+ and A^- both become global attractors at $\epsilon=0.34$, which lies slightly above ϵ_1, their basins of attraction are not intermingled. Indeed, as shown in Fig. 10, both basins of attraction are solid regions of either black points or white points. This demonstrates our notion that intermingled basins do not occur if the on-off intermittent trajectory on the other side of the transition point exhibits no communication between the two restricted attractors.

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